Lecture-9

- One dimensional solution of Laplace’ Equation in cylindrical coordinate system

One dimensional solution of Laplace' Equation in cylindrical coordinate system

Therefore we move over to the cylindrical coordinate system for our next example.

Again, Variations with respect to z are nothing new, the same as seen in rectangular coordinates (last example), and hence, in cylindrical coordinates, we consider variations in ρ and ϕ .

We have, in cylindrical cooridinates, the Laplace' equation as

$$\nabla^2 \mathbf{V} = \frac{\mathbf{1}}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \quad \mathbf{(}Cylindrical\mathbf{)}$$

We consider that V is a function of ρ only. In this case the Laplace' equation in cylindrical coordinates reduces to

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)=0$$

Since ρ is in the denominator, we exclude ρ = 0 from our solutions. Then we multiply throughout by ρ and get

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)=0$$

Integrating this equation once we get,

$$\rho\frac{\partial V}{\partial\rho}=A, \quad or \quad \frac{\partial V}{\partial\rho}=\frac{A}{\rho}$$, A an arbitrary constant

Integrating once again, we get

$$V = A\ln\rho + B,$$ B an arbitrary constant

From this equation, we observe that equipotential surfaces are given by ρ = constant and are cylinders. Example of the problem is that of a coaxial capacitor or coaxial cable.

Let us create the boundary conditions by choosing $V = V_a$ at $\rho = a$ and $V = 0$ at $\rho = b$, $b > a$.

Then we get from the above equation,

$$V = V_a = A\ln a + B$$

$$V = V_b = A\ln b + B$$

Solving these two equations for A and B we get

$$A = \frac{V_a - V_b}{\ln\left(\frac{a}{b}\right)} \quad and \quad B = \frac{V_b \ln a - V_a \ln b}{\ln\left(\frac{a}{b}\right)}$$

Substituting these the values of A and B in the general Expression for V , we get

$$V = \frac{V_a - V_b}{\ln\left(\frac{a}{b}\right)} \ln \rho + \frac{V_b \ln a - V_a \ln b}{\ln\left(\frac{a}{b}\right)}$$

Letting $V_b = 0$, we get

$$V = V_a \frac{\ln\left(\frac{b}{\rho}\right)}{\ln\left(\frac{b}{a}\right)}$$

Next we find the capacitance of the system using the procedure we have used for the parallel plate capacitor as

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial \rho} = -\frac{\partial}{\partial \rho}\left(\frac{V_a}{\ln\left(\frac{b}{a}\right)}\ln\frac{b}{\rho}\right)\hat{a}_\rho = -\frac{V_a}{\ln\left(\frac{b}{a}\right)}\frac{\partial}{\partial \rho}\left(\ln\frac{b}{\rho}\right)\hat{a}_\rho$$

i.e., $$\vec{E} = \frac{V_a}{\rho} \frac{1}{\ln\left(\frac{b}{a}\right)} \hat{a}_\rho$$

Hence $$\vec{D} = \varepsilon \, \vec{E} = \varepsilon \frac{V_a}{\rho} \frac{1}{\ln\left(\frac{b}{a}\right)} \hat{a}_\rho = \vec{D}_S$$

Then

$$\vec{D}_S = \varepsilon \frac{V_a}{a} \frac{1}{\ln\left(\frac{b}{a}\right)} \hat{a}_\rho = D_S \hat{a}_N = D_N \hat{a}_N \quad and$$

Therefore $$D_N = \rho_S = \frac{\varepsilon}{a} V_a \frac{1}{\ln\left(\frac{b}{a}\right)}$$

Therefore,

$$Q = \int_{\substack{S \text{ of one of} \\ \text{the plates} \\ \text{with } \rho=a}} \rho_S \, dS = \rho_S \int_{\substack{S \text{ of one of} \\ \text{the plates} \\ \text{with } \rho=a}} dS$$

$$= \frac{\varepsilon}{a} V_a \frac{1}{\ln \frac{b}{a}} S$$

$$= \frac{\varepsilon}{a} V_a \frac{1}{\ln \frac{b}{a}} 2 \pi a L$$ over a length L meters of the coaxial cable

$$\therefore \; C = \frac{|Q|}{V_a} = \frac{2 \pi \varepsilon L}{\ln \frac{b}{a}}$$

Therefore the capacitance per unit length of the line C_L is

$$\therefore \; C_L = \frac{2\pi\varepsilon}{\ln \frac{b}{a}} \quad \text{F/m}$$

Next we consider V as a function of ϕ only. In this case The Laplace' equation in cylindrical coordinate system reduces to

$$\nabla^2 \mathbf{V} = \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

i.e.,

$$\frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0$$

Excluding the value ρ = 0 this equation becomes

$$\frac{d^2V}{d\phi^2} = 0$$

Integrating both sides we get

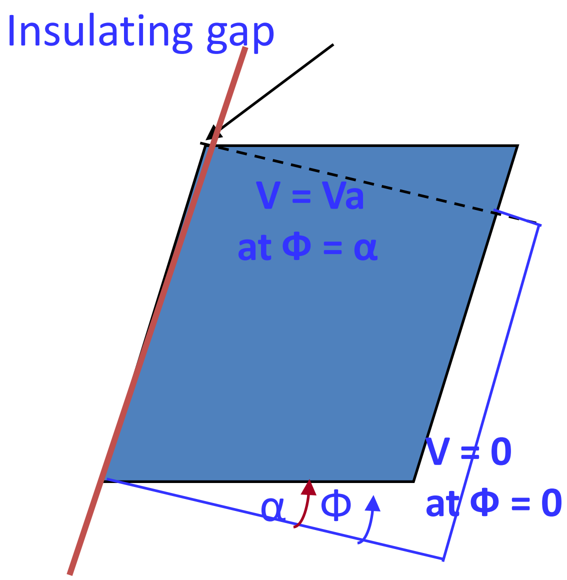
$$\frac{dV}{d\phi} = A$$

Integrating once again, we get

$$V = A\phi + B$$

This is the general equation for V when V is a function of only ϕ

From this equation, we observe that equipotential surfaces are given by Φ = constant and are planes To visualize this, let us choose Let us choose two such equipotential surfaces, $V = V_a$ at $\Phi = \alpha$ and $V = 0$ at $\Phi = 0$. An example of the problem is that of a corner reflector antenna, a very useful antenna system in communication systems.



For the chosen boundary condition, we get

$$V = V_a \quad = A\alpha + B$$

$$V = 0 \quad = A\,0 + B \qquad \therefore B = 0$$

and $$A = \frac{V_a}{\phi}$$

Thus the general expression for V becomes

$$V = \frac{V_a}{\alpha}\phi$$

Once again we follow the 5 step procedure to determine the capacitance of the system

$$\vec{E} = -\nabla V = -\frac{1}{\rho}\frac{\partial V}{\partial \rho} = -\frac{1}{\rho}\frac{V_a}{\alpha}\hat{\rho}$$

Note that E is a function of ρ and not of Φ. But the vector field **E** Is a function of Φ. Now,

$$\vec{D} = \varepsilon\vec{E} = -\frac{\varepsilon}{\rho}\frac{V_a}{\alpha}\hat{\rho}$$

$$\therefore \vec{D}_S = D_S\hat{a}_S = D_N\hat{a}_N = -\frac{\varepsilon}{\rho}\frac{V_a}{\alpha}\hat{\rho}$$

$$\therefore D_N = -\frac{\varepsilon}{\rho}\frac{V_a}{\alpha} = \rho_S$$

The surface integration on ρ_S gives Q:

$$Q = \oint_S \rho_S dS = \oint_S -\frac{\varepsilon}{\rho}\frac{V_a}{\alpha} ds = -\frac{\varepsilon}{\rho}\frac{V_a}{\alpha}\oint_S dS$$

and we get immediately the value for the capacitance of the corner reflector system as

$$C = \frac{|Q|}{V_a} = \frac{\frac{\varepsilon}{\rho}\frac{V_a}{\alpha}\oint_S dS}{V_a} = \frac{\varepsilon}{\rho\alpha}\oint_S dS$$